J-holomorphic curves: $(\Sigma, j)$ Riemann surface, $(M,\omega)$ symplectic + compatible $J$

**Def:** $u : (\Sigma, j) \longrightarrow (M, J)$ is J-holomorphic if $du \circ j = J \circ du$, or

$$\bar\partial_J u := \frac{1}{2}\left(du + J \circ du \circ j\right) = 0 \in \Omega^{0,1}(\Sigma, u^*TM)$$

complex antilinear 1-forms on $\Sigma$ w/ vals. in the complex vector bundle $u^*TM$.

We study the moduli space of J-holom. curves

- with fixed domain = Riem. surface (possibly with marked pts $z \in \Sigma$)
  - (or with fixed topological type but let $j$ vary) (poss. with boundary, w/ Lagr. boundary cond$^{n}$)
- & representing a fixed homology class in $H_2(M)$

up to reparametrization equivalence: $u \sim u'$ if $\exists$ biholom. $\phi$

$$\begin{tikzcd} (\Sigma, j, z) \arrow[d, "\phi"', "\simeq"] \arrow[dr, "u"] & \\ (\Sigma', j', z') \arrow[r, "u'"'] & M \end{tikzcd}$$

(if fixed domain, $\phi \in \mathrm{Aut}(\Sigma, j)$).

- Start with the simplest case of fixed domain $(\Sigma, j)$, for now a closed Riem. surface

Local structure of the moduli space is governed by the linearized $\bar\partial_J$ operator:

given $u : \Sigma \to M$ J-holomorphic,

the tangent space to maps $\Sigma \to M$ is given by vector fields along $u$,

$v \in C^\infty(\Sigma, u^*TM)$, or better, $W^{k,p}(\ldots)$ = k derivatives in $L^p$ - Banach space!

(typically take $k=1$, $p>2$ so that $k - \frac{2}{p} > 0 \Rightarrow W^{k,p} \hookrightarrow C^0$)

$$D_u : W^{k,p}(\Sigma, u^*TM) \longrightarrow W^{k-1,p}(\Sigma, T^*\Sigma^{0,1} \otimes u^*TM)$$

ie complex antilinear $(T_z\Sigma, j) \to (T_{u(z)}M, J)$

$$D_u(v) = \nabla_{\frac{\partial}{\partial t}|_{t=0}} \bar\partial_J\left(\exp_u(tv)\right) = \frac{1}{2}\left(\nabla v + J \circ \nabla v \circ j + (\nabla_v J) \circ du \circ j\right)$$

(pullback of) Levi-Civita connection for J-compatible $g$

$\hookrightarrow \nabla^{LC} J = 0$ only in Kähler case!

(This involves connections because need to compare (0,1)-forms in $u^*TM$ vs. in $\exp_u(tv)^*TM$!!

do so using connection induced by Levi-Civita).

But... when $u$ is J-holomorphic, formula is indep$^t$ of choice of $\nabla$ (because: $\nabla s = \pi^\nabla(ds)$

splitting is indep$^t$ of $\nabla$ along $s^{-1}(0)$).

This is a real linear Cauchy-Riemann operator on sections of the complex vector bundle $u^*TM \downarrow \Sigma$

(complex linear when $J$ integrable: then $D = \bar\partial$ operator on holomorphic bundle $u^*TM \to \Sigma$).

ie. in local holom coordinates on $\Sigma$ + local trivialization of $u^*TM$, it is of the form

$$Dv = \bar\partial v + (Av)^{0,1} \quad \text{where } A \in W^{k-1,p}(\Sigma, T^*\Sigma \otimes \mathrm{End}_{\mathbb{R}}(u^*TM)).$$

$D$ is an elliptic diff$^l$ operator, ie. its principal symbol (= leading order action on Fourier transform) is invertible. This gives estimates $\|v\|_{W^{k,p}} \leq C\left(\|Dv\|_{W^{k-1,p}} + \|v\|_{W^{k-1,p}}\right)$.

Thm: D is a Fredholm operator, ie. has finite dim! kernel & cokernel. Its index is given by the Riemann-Roch formula:

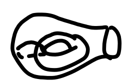
$$\mathrm{ind}_{\mathbb{R}}(D) = \dim_{\mathbb{R}} \ker D - \dim_{\mathbb{R}} \mathrm{Coker}\, D = \overset{=\dim_{\mathbb{C}} M}{n}\, \chi(\Sigma) + 2\langle c_1(u^*TM), [\Sigma]\rangle$$
$$= n(2-2g) + 2\langle c_1(TM), [u(\Sigma)]\rangle.$$

Rmk: • $\mathrm{Im}\, D = \ker(D^*)^{\perp}$  $D^*$ formal adjoint (wrt hermitian metric on $u^*TM$, eg. compatible g)
$$D^*: W^{k,p}(\Sigma, \Lambda^{0,1}\otimes u^*TM) \to W^{k-1,p}(\Sigma, u^*TM)$$
$\leftrightarrow$ via complex conjugation, a Cauchy-Riemann operator on $T^*\Sigma \otimes u^*\overline{TM}$.

• elliptic regularity $\Rightarrow$ $\ker D, \ker D^* \subset C^\infty$ (if $J$ $C^\infty$, else as regular as $J$).

• compare with usual Riemann-Roch formula, noting that
  → Fredholm index is invariant under homotopy hence order 0 terms in $D_u$ don't matter.
  → real index = 2·(complex index)
  → $\mathrm{rank}_{\mathbb{C}}\, n$ vect. bundle

Why this matters: assume $D_u$ is onto for all J-holom. $u: \Sigma \to M$ in a given homology class A. (Say J regular). Then $\tilde{\mathcal{M}} = \{u: \Sigma \to M \,/\, \bar\partial_J u = 0,\ [u] = A\}$ is a smooth mfld of $\dim_{\mathbb{R}} = $ index.
so far: fixed domain, not quotienting by $\mathrm{Aut}(\Sigma)$.  with $T_u\tilde{\mathcal{M}} \equiv \ker D_u$.
This follows from transversality / implicit function theorem

First Chern class: $E \to \Sigma$ complex vector bundle over a closed (oriented) surface, rank n.
$\mathring{\Sigma} = \Sigma - \text{disc}$ retracts onto $\vee S^1$'s  , over which E is trivial.

(Indeed, gluing data over each $S^1 =$  $\mathbb{C}^n \times [0,1] / (v,0) \sim (\phi(v),1)$  $\phi \in GL_n\mathbb{C}$
$\pi_0 GL_n(\mathbb{C}) = 1 \Rightarrow \exists\, \varphi_t \in GL_n,\ \varphi_0 = id,\ \varphi_1 = \phi$. Then $(v,t) \mapsto (\varphi_t(v), t)$ gives isom. with trivial bundle). [Ex: do this in smooth category, gluing over interval by smooth map]

Now glue $\Sigma = \Sigma^0 \cup_{S^1} D^2$ $\Rightarrow$ trivial over both pieces, gluing is a map $S^1 \to GL_n(\mathbb{C})$ and what matters is its homotopy class in $\pi_1(GL_n(\mathbb{C}))$ [if loop extends over $D^2$ then can use it to modify trivialization on $D^2$ side & show the bundle is globally trivial].

Hence: rank n $\mathbb{C}$ vect bundles over $\Sigma$ are classified by their degree $\in \pi_1(GL_n\mathbb{C}) \cong \mathbb{Z}$.
isom. is induced by $\det: GL_n(\mathbb{C}) \to \mathbb{C}^*$

In general, $E \to M$ cx. vector bundle $\Rightarrow$ first Chern class $c_1(E) \in H^2(M,\mathbb{Z})$ characterized by: $\forall\, \Sigma \xrightarrow{f} M$, $\langle c_1(E), \underset{H_2(M,\mathbb{Z})}{\underset{\cap}{[\Sigma]}}\rangle = \deg(f^*E \to \Sigma) \in \mathbb{Z}$.

By constr. this is functorial under pullbacks, additive under $\oplus$, and $c_1(E) = c_1(\det E := \Lambda^n E)$.
(but doesn't fully classify higher rank $\mathbb{C}$ vect bundles over higher-dim! manifolds).

* The case where $\Sigma$ has boundary works equally well, if we impose a totally real boundary condition: eg. require $u(\partial\Sigma) \subset L$ some Lagr. submanifold of $M$.

Then $u^*TL \subset (u^*TM, J)$ is a totally real subbundle of the $\mathbb{C}$-vect bundle $u^*TM$ over $\partial\Sigma$, and wrt compatible metric, $J(TL) = TL^\perp$. The linearized op. is now

$$D_u : W^{k,p}(\Sigma, u^*TM, u^*TL) \longrightarrow W^{k,p}(\Sigma, \Lambda^{0,1} \otimes u^*TM)$$
$$= \{ v \in W^{k,p}(\Sigma, u^*TM) \,/\, v_{|\partial\Sigma} \in u^*TL \} \qquad \text{(same formula as before)}$$

is again Fredholm, with $\mathrm{ind}(D_u) = n\chi(\Sigma) + \underbrace{\mu(u^*TM, u^*TL)}_{\text{Maslov index}}$

<u>Maslov index</u>: $u^*TM$ is trivial as cx. vector bundle, so the relevant piece of information is the loop of Lagrangian subspaces $u^*TL \subset u^*TM \cong \mathbb{C}^n$ along $\partial\Sigma$ (or loops if several $\partial$ components)

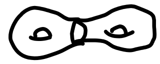
$\pi_1 LGr(n) = \pi_1(U(n)/O(n)) \cong \mathbb{Z}$, induced by $\det^2 : U(n) \to S^1$.

(or, for totally real, $GL(n,\mathbb{C})/GL(n,\mathbb{R})$ - same).

Alternatively, Maslov index of a loop of Lagr. planes $\subset \mathbb{R}^{2n}$ $\equiv$ count (w/ signs & multiplicities) the number of times $L(t)$ fails to be transverse to a given Lagrangian $L_0$.

Can also think of $\mu$ as (twice) a <u>relative</u> first Chern class rel. boundary. Ie. if the subbundle $u^*TL \to \partial\Sigma$ is orientable hence trivial, pick a trivialization to get a triv$^n$ of $u^*TL \otimes \mathbb{C} = u^*TM$ over $\partial\Sigma$, then can define a rel. first Chern class by comparing this with a triv$^n$ over $\Sigma$.

($\Leftrightarrow$ extend trivially over capping disc to $\bar{\Sigma} = \Sigma \cup D^2$ & calc. usual $c_1$) Then $\mu = 2c_1$.

* Can think of index formula with boundary as: given $\Sigma$, build <u>double</u> $\hat{\Sigma} = (\Sigma, j) \cup_{\partial} (\bar{\Sigma}, -j)$ by reflection about $\partial\Sigma$  and glue $E = u^*TM \to \Sigma$ to $\bar{E} = u^*\overline{TM} \to \bar{\Sigma}$ over $\partial\Sigma$ by reflection about Lagr. subspaces $F = u^*TL$, ie. $E \to \bar{E}$ def$^d$ as $1$ on $F$ and $-1$ on $JF$.

Then get $\hat{E} \to \hat{\Sigma}$ with $c_1(\hat{E}) = \mu(E,F)$. Up to $0^{th}$ order terms which don't affect Fredholm index, $\bar{\partial}$ operator on $\hat{E} \to \hat{\Sigma}$ commutes with complex conjugation exch. the two halves. $\pm 1$ eigenspaces = doubled sections of $E$ w/ boundary in $F$, resp. $JF$. $\Rightarrow$ $\mathrm{ind}\, D_{(\Sigma,E,F)} = \frac{1}{2}\, \mathrm{ind}\, \hat{D}_{(\hat{\Sigma},\hat{E})}$.

<u>Ex</u>: $u : D^2 \xrightarrow[\text{incl.}]{} (\mathbb{R}^2, S^1)$  $TL$ $\quad \mu = 2 \Rightarrow \mathrm{ind}(D_{\bar{\partial}}) = n + \mu = 1 + 2 = 3$.

$T_u \tilde{\mathcal{M}} = \ker D_u$ 3-dim$^l$, in this case it just corresponds to reparam. of $u$ by $\underbrace{\mathrm{Aut}(D^2)}_{\dim_{\mathbb{R}} = 3} = PSL(2,\mathbb{R})$. $\leftarrow$ autom. of $\mathbb{CP}^1$ are $PSL(2,\mathbb{C})$, $\frac{az+b}{cz+d}$. Those fixing $\mathbb{RP}^1$ (& not swapping hemispheres) $= PSL(2,\mathbb{R})$

(Riem. mapping theorem $\Rightarrow$ up to reparam. $\exists!$ holom. $u$, $\mathcal{M} = \tilde{\mathcal{M}}/\mathrm{Aut} = \{pt\}$)

* In general: $\mathrm{Aut}(S^2) = PSL(2,\mathbb{C})$ cx. dim. 3    $S^2$ w/ 1 or 2 marked pts still has autom's.
acts singly transitively on triples of pts.

$\mathrm{Aut}(T^2 = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})$ = finite extension of $T^2$, cx. dim. 1 ; $\mathcal{M}_1 = \{\tau \in \mathbb{H}/PSL_2\mathbb{Z}\}$ $\dim_{\mathbb{C}} 1$

all others have discrete Aut (generically $\{id\}$), and $\mathcal{M}_g = \{(\Sigma, j)\}/\sim$ has $\dim_{\mathbb{C}} 3g-3$.

With $k$ marked points: in <u>stable case</u> $\begin{pmatrix} g\geq 2 & \\ g=1 & k\geq 1 \\ g=0 & k\geq 3\end{pmatrix}$ $\dim_{\mathbb{C}} \mathcal{M}_{g,k} = 3g-3+k$.

$$\mathcal{M}_{g,k}(M, J; \underset{H_2(M,\mathbb{Z})}{\underset{\cap}{A}}) = \Big\{ u : \underbrace{(\Sigma, j, z)}_{\text{genus } g,\ k \text{ marked pts}} \longrightarrow M \,/\, \bar\partial_J u = 0,\ u_*[\Sigma] = A \Big\}/\sim$$

first order deformations now come from

$\begin{cases} v \text{ deformation of map } u = \text{section of } u^*TM \\ \text{deformation of } j \in \text{some finite dim}^{l} \text{ subspace of } T_j\mathcal{J}_\Sigma = \{j' \in \mathrm{End}(T\Sigma) \,/\, jj'+j'j = 0\} \end{cases}$
$= \Omega^{0,1}(\Sigma, T\Sigma)$

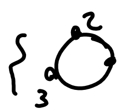
which is a local transverse slice to action of $\mathrm{Diff}(\Sigma)$.

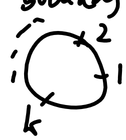
linearized operator $D_{(u,j)}(v, j') = D_u(v) + \frac{1}{2} J \circ du \circ j' \in \Omega^{0,1}(\Sigma, u^*TM)$.

If $J$ is <u>regular</u> ie. $D_{(u,j)}$ surjective at all solution, then $\mathcal{M}_{g,k}(M,J,A)$ is smooth of

$$\dim_{\mathbb{R}} = d := \underbrace{n(2-2g) + 2c_1(TM)\cdot A}_{\mathrm{ind}(D_u)} + \underbrace{2(3g-3+k)}_{\dim \mathcal{M}_{g,k}} = (2n-6)(1-g) + k + 2c_1(TM)\cdot A.$$

(or $-\dim \mathrm{Aut}(\Sigma, z)$ in unstable case)

Similarly in case with boundary: $\mathrm{Aut}(D^2) = PSL(2,\mathbb{R})$ acts transitively on $\Big\{$  $\Big\}$, and when $\ell \geq 3$ the moduli space of ($D^2$, $\ell$ boundary pts) is a contractible $(\ell-3)$-dim$^{l}$ manifold.




(place first three points at $1, i, -1$, then $\ell-3$ pts in order along semicircle...)

$\mathcal{M}_{k,\ell}(M, L; J, A)$    expected $\dim_{\mathbb{R}}$ $d = n-3+2k+\ell+\mu(A)$.

discs w/ boundary in $L$,
$k$ interior & $\ell$ boundary marked pts
representing rel. class $A$

(in this case, easier to think of just moving the points than deforming $j$ !)   (other $\Sigma$: $(n-3)\chi(\Sigma)+\ldots$)

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* <u>Automatic regularity</u>: consider the case where $\Sigma = S^2$ and $J$ integrable.

Then $D = \bar\partial$ operator on $u^*TM$ holomorphic v.b. over $\Sigma = \mathbb{P}^1$.

A result of Grothendieck $\Rightarrow$ $u^*TM = \bigoplus$ line bundles $\mathcal{O}(d_i)$, so $\ker \bar\partial = \bigoplus H^0(\mathbb{P}^1, \mathcal{O}(d_i))$
$\mathrm{Coker}\, \bar\partial = \bigoplus H^1(\mathbb{P}^1, \mathcal{O}(d_i))$

But $\bar\partial$ operator is surjective on $\mathcal{O}(d_i)$ iff $d_i \geq -1$

Hence: $u$ is regular <u>iff</u> all $d_i \geq -1$.

even in nonintegrable case, if somehow $u^*TM = \bigoplus$ line bundles st. $D_u$ splits. (or just triangular!) we can often conclude surjectivity!

⑤

$D$ Cauchy-Riemann operator on a complex line bundle $L \to \Sigma$ Riem. surface:

$\deg(L) < 0 \Rightarrow \ker D = 0$. Indeed, $Ds = 0$ for $s \not\equiv 0 \Rightarrow$ the zeros of $s$ are isolated and have positive multiplicity (Carleman similarity principle! Note $Ds = \bar\partial s + As = 0 \Rightarrow \bar\partial s = -As$ in loc. trivialization)

& considering $D^* \Leftrightarrow \bar\partial$-operator on $T^*\Sigma \otimes L^*$: $\deg(L) > 2g-2 \Rightarrow \operatorname{Coker} D = 0$.

Ex: $u: S^2 \to M^4$ J-holomorphic & embedded $\Rightarrow TS^2 = O(2) \subset u^*TM$ preserved by $D$, regularity holds iff the normal bundle has degree $\geq -1$ (ie. $[u(S^2)]\cdot[u(S^2)] \geq -1$). ($\Leftrightarrow c_1(TM)\cdot[u(S^2)] \geq 1 \Leftrightarrow$ expected dim $\mathcal{M} \geq 0$).

Similarly for discs, $u: D^2 \to (M,L)$: if $u^*TM$ splits into $\oplus$ complex line bundles $\cup$ $u^*TL$ $\cup$ $\oplus$ real subbundles and $D_u$ splits, then regularity holds iff each summand has $\mu \geq -1$.

Ex: $u: D^2 \to (M^4, L)$ J-holom. & embedded $\Rightarrow$ regularity holds iff Maslov index $\mu \geq 1$, ie. exp$^d$ dim $\geq 0$.
$(TD^2, TS^1) \subset (u^*TM, u^*TL)$ has $\mu = 2$, need normal bundle to have $\mu \geq -1$.

★ <u>Generic J & regularity for somewhere injective curves:</u>

Say $u: \Sigma \to M$ is somewhere injective if $\exists\, p \in \Sigma$ st. $du(p) \neq 0$ & $u^{-1}(u(p)) = \{p\}$.

<u>Fact:</u> for $\Sigma$ closed Riem. surf., either $u$ factors through a covering map $\Sigma \xrightarrow{\pi} \Sigma' \xrightarrow{v} M$ & nowhere inj., then say $u$ is <u>multiply covered</u>; or $u$ is injective at all but finitely many points, say $u$ <u>simple</u>.

In case w/ boundary this isn't quite true, because $u(\Sigma)$ can self-overlap partially (or wholly) without being multiply covered.

Let $\mathcal{M}^* \subset \mathcal{M}$ subset: somewhere injective curves $u: \Sigma \to M$.

<u>Thm:</u> $\exists$ Baire dense set $\mathcal{J}_{reg} \subset \mathcal{J}(M,\omega)$ st., if $J \in \mathcal{J}_{reg}$ and $u$ is J-holomorphic & somewhere injective, then $D_u$ is onto. ($C^\ell$ topology for some $\ell \geq 1$)

(So: for $J \in \mathcal{J}_{reg}$, somewhere inj. curves are regular, and $\mathcal{M}^*(J,A)$ is a manifold of the expected dim.)

To show this, consider universal moduli space
$$\widetilde{\mathcal{M}}^* = \left\{ (\Sigma, j, u: \Sigma \to M, J) \,\middle/\, \begin{array}{l} J \in \mathcal{J}(M,\omega) \\ \bar\partial_J u = 0 \\ u_*[\Sigma] = A \\ u \text{ somewhere inj.} \end{array} \right\} \Big/ \sim \;=\; \coprod_{J \in \mathcal{J}} \mathcal{M}^*(J,A)$$
(somewhere injective curves only)

& projection to $\mathcal{J}(M,\omega)$

First order deform$^n$ of $\bar\partial$-equation allowing $J$ to vary is given by

given by $\widetilde{D}_{(u,j,J)}(v, j', J') = D_u(v) + \frac{1}{2} J \circ du \circ j' + \frac{1}{2} J'(u) \circ du \circ j$.

$v \in W^{k,p}(\Sigma, u^*TM)$; $j'$ (subspace of $T_j \mathcal{J}_\Sigma$) if allow domain deform$^n$; $J' \in T_J\mathcal{J} = \{ J' \in C^\ell(M, \operatorname{End} TM) \,/\, JJ' + J'J = 0,\ \omega(J'\cdot,\cdot) = -\omega(\cdot, J'\cdot) \}$ $(\ell \geq k)$

<u>Prop.:</u> if $u$ somewhere inj., then $\widetilde{D}$ is always surjective!

PF: $\mathrm{Im}\,\tilde{D}_{(u,j,J)} \supset \mathrm{Im}\,D_u$ is closed of finite codim $\Rightarrow$ enough to show $(\ )^\perp = 0$.

Assume $\eta \in L^q(\Sigma, \Lambda^{0,1}\otimes u^*TM)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ s.t.

$\eta \in \ker D_u^* = (\mathrm{Im}\,D_u)^\perp$ and $\eta \perp \{J'(u)\,du\,j \;/\; J' \in T_J\mathcal{J}\}$ $\Rightarrow$ want $\eta \equiv 0$?

for pairing $\int_\Sigma \langle\cdot,\cdot\rangle$

Already know: $\eta \in \ker D_u^* \Rightarrow \eta$ as regular as $J$ and, if $\eta \not\equiv 0$, has isolated zeroes.

↳ Cauchy-Riemann operator

Claim: $\eta \perp \{J'(u)\,du\,j\} \Rightarrow \forall z \in \Sigma$ injective point, $\eta(z)=0$.

(i.e. $du(z)\neq 0$, $u^{-1}(u(z)) = \{z\}$)

Indeed: assume $z$ injective point. linear alg. in $T_{u(z)}M \Rightarrow \exists J'$ st.

$J'_{u(z)}$ maps $\mathrm{Im}\,du(z)$ to any complex line in $T_{u(z)}M$ (in cx. antilinear manner).

Then if $\eta(z)\neq 0$ we can build $J'$ st. $\langle \eta(z), J'(u(z))\circ du(z)\circ j\rangle \neq 0$ at $z$.

Use cutoff functions $\Rightarrow$ can build $J'$ with support in a small neighborhood of $u(z)$ which only intersects the curve $u$ in a small nbd of $z$. Then get $\int\langle\,,\,\rangle \neq 0$. □

This yields the proposition, since the zeroes of $\eta \in \ker D_u^*$ are isolated unless $\eta\equiv 0$, but injective points are an open subset of $\Sigma$. □.

Hence: $\tilde{\mathcal{M}}^*$ is a Banach manifold, and the projection $\tilde{\mathcal{M}}^* \xrightarrow{\pi} \mathcal{J}$ is a Fredholm map of index $d=$ expected dim. $\mathcal{M}$.

Sard-Smale theorem: the set $\mathcal{J}^{reg}$ of regular values of $\pi$ is a Baire dense subset of $\mathcal{J}$.

Transversality: if $J \in \mathcal{J}^{reg}$ and $(u,j) \in \mathcal{M}^*(A,J)$ somewhere inj. $J$-holom curve, then

$$d\pi : T_{(u,j,J)}\tilde{\mathcal{M}}^* = \ker \tilde{D}_{(u,j,J)} \to T_J\mathcal{J}^* \text{ onto (since } J \text{ reg. value of } \pi) \quad (*)$$

But also: $\tilde{D}_{(u,j,J)} : (v,j',J') \longmapsto D_{(u,j)}(v,j') + \frac{1}{2}J'\,du\,j$ onto (by prop.)

so $\forall \eta \in W^{k-1,p}(\Sigma, \Lambda^{0,1}\otimes u^*TM)$, $\exists (v,j',J')$ st. $\tilde{D}_{(u,j,J)}(v,j',J') = \eta$

$(*) \Rightarrow \exists (v_1,j_1')$ st. $\tilde{D}_{(u,j,J)}(v_1,j_1',J') = 0$ $\Big\} \Rightarrow D_{(u,j)}(v-v_1, j'-j_1') = \eta$

Hence $D_{(u,j)}$ is onto $\forall (u,j) \in \mathcal{M}^*(A,J)$. □.

Similar transversality argument $\Rightarrow \forall J_0, J_1 \in \mathcal{J}^{reg}$, $\exists J_{t\,(t\in[0,1])}$ s.t. $\bigsqcup_{t=[0,1]} \mathcal{M}^*(A,J_t) = \pi^{-1}(\text{path } J_t) \subset \tilde{\mathcal{M}}^*$ is a smooth mfld of dim. $d+1$.

($\rightsquigarrow$ can compare different regular $J$'s).

Hovever: regularity always fails for multiply covered maps in the wrong range of $c_1/\mu$.

Ex: $\Sigma = S^2$, $A = mA_1$ $\Rightarrow$ expected dim. $\mathcal{M}(J,A) = 2n-6+2c_1(M)\cdot A$

vs. multiply covered maps: $u = u_1 \circ f : S^2 \xrightarrow[m:1]{f} S^2 \xrightarrow{u_1} M$

$\hookrightarrow$ dim. $2n-6+2c_1(M)\cdot A_1$ if regular

$\hookrightarrow$ ratio of two deg. $m$ polynomials ($\dim_{\mathbb{C}}$ $2m+1$)
$2m-2$ ramification points — up to reparam. $\Rightarrow 2m-2$.

so if $\mathcal{M}(J,A_1)$ is regular, get $\mathcal{M}^{\text{mult cover}}(J,A)$ dim. $2n-6+2c_1(M)\cdot A_1 + 4m-4$
This is $> 2n-6+2m\,c_1(M)\cdot A_1$ as soon as $c_1(M)\cdot A_1 < 2$.

Similarly, multiple covers of $J$-holom. discs of Maslov index $<2$ are never regular.
($n-3+m\mu < n-3+\mu+2m-2$ for $\mu<2$).

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* Many large pieces of machinery to deal with non-regular moduli spaces – the hardest issue is regularity for multiply covered components of stable curves (= compactification of $\mathcal{M}$, see later).

- in alg. geometry ($\Sigma$ closed surface, $J$ integrable), obstruction sheaf $Obs \to \mathcal{M}(A,J)$
with fiber $Obs_u = H^1(\Sigma, \mathcal{N}_u)$, $\mathcal{N}_u = u^*TM/T\Sigma$. $[\mathcal{M}^{\text{virt}}] = e(Obs) \in H^*(\mathcal{M})$ Euler class
(Jun Li, Behrend-Fantechi, ...)

 Heuristics: Assume $\mathcal{M}$ smooth of wrong dim., with rank Coker $D = r$ constant.
 Then $Obs$ is a rank $r$ vector bundle over $\mathcal{M}$.
 If we perturb $\bar\partial_J u = 0$ equation (eg. to $\bar\partial_{J+\delta J'} u = 0$ or $\bar\partial_J u = t\nu(u)$)
 then projecting $\nu(u)$ to Coker $D_u$ gives a section of $Obs \to \mathcal{M}$, and
 $u \in \mathcal{M}$ deforms to first order iff $\exists v$ st. $D_u(v) = \nu(u)$, ie. $\pi_{\text{Coker}}\nu(u) = 0$.
 Thus: expect perturbed sol. space $\cong$ zero set of a section of $Obs \to \mathcal{M}$
 This corresponds to a cycle in $\mathcal{M}$ dual to the Euler class $e(Obs)$.

- various approaches in symplectic geometry: "virtual fundamental classes", multivalued perturbations, ...
(Fukaya-Ono, Li-Liu-Ruan-Tian, Hofer-Salamon, ...). More recently,

 → concrete: achieve regularity using a domain-dependent acs. $J$ on $M$ which depends on the domain $(\Sigma, j, z)$ and on the point of $\Sigma$. The symmetry of multiply covered maps is broken by deforming $J$ differently in the different sheets of the cover $\Sigma \to \Sigma_1$. This requires breaking domain automorphisms by having enough marked points. Eg can place "stabilizing" marked pts at all intersections of $u(\Sigma)$ with a very high degree cx. hypersurface. (Cieliebak-Mohnke).

→ Kuranishi structures (= $\mathcal{M}$ covered by local charts $\simeq s_i^{-1}(0) \subset U_i$, $s_i$ section of a vector bundle over $U_i$ – not assumed transverse to zero section).
(FOOO for Lag. Floer theory; cleanup work by McDuff-Wehrheim; Joyce; Pardon "implicit atlases")

→ polyfolds (Hofer-Wysocki-Zehnder): new functional analysis foundations ("scale calculus"), (compactified) space of maps to $M$ is a polyfold, $\bar\partial_J$ sc-smooth section of a strong polyfold bundle, abstract Fredholm perturbation theory gives a regularized moduli space.